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On the Existence of Equilibrium for Abstract Economies*

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In this paper, several existence theorems for abstract economies have been established. They are natural modifications (strict generalizations when strategy spaces are metrizable) of the main results of Yannelis and Prabhakar [23], Tulcea [22], and Ding, Kim, and Tan [10] for infinite-dimensional casas. For the finite-dimensional case, we show that the topological conditions on correspondences in Shafer and Sonnenschein's result [20] can be much weaker. Along the proofs, several propositions are established, which are of independent interest. We also show that many existence results in fixed point theory, maximization theory, and generalized quasi-variational inequalities can be encompassed.

1. INTRODUCTION

In [9], Debreu proved an existence theorem for social equilibrium—the classical Arrow–Debreu–McKenzie model of exchange under perfect competition. This result is in turn a generalization of the Nash [18] noncooperative equilibrium result. The basic features of the classical model are twofold:

(1) its finiteness, i.e., both the set of agents and the number of commodities are finite;

(2) agents behave in a transitive, complete and continuous fashion, i.e., agents' preferences are assumed to be transitive and complete and consequently are representable by continuous utility functions.

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Since then three major extensions of the Arrow–Debreu–McKenzie model have been made. The first is the extension of the set of agents to a measure space of agents by Aumann [3, 4]. The second major generalization is due to Bewley [7], which permits the commodity space to be infinite-dimensional. This generalization allows people to deal with problems involving infinite time horizons, uncertainty about an infinite number of states of the world, or infinite varieties of commodity characteristics. The third major extension is due to a significant contribution made by Mas-Colell [15], who shows that even for preferences are not transitive or complete, an equilibrium still exists. With introducing a theoretic game setting called abstract economy, the above results have further been improved by Shafer and Sonnenschein [20], Borglin and Keiding [8], and Yannelis and Prabhakar [23], recently by Tulcea [22], and Ding, Kim, and Tan [10].

On the other hand, (generalized) quasi-variational inequalities of various types have been extensively studied in the past years (see, e.g., [5, 17]) with different kinds of approaches. With the theoretic setting of abstract economy, it is clear that (generalized) quasi-variational inequalities are very special cases of abstract economy. So many different approaches in (generalized) quasi-variational inequalities can be unified.

2. Notation and Definitions

Let X and Y be two topological spaces. Throughout this paper, we denote a function f from X into Y by $f: X \to Y$ and a correspondence (set-valued mapping) F from X into Y by $F: X \rightsquigarrow Y$. A correspondence $F: X \rightsquigarrow Y$ is said to be *upper semi-continuous* (in short, u.s.c.) if the set $\{x \in X : F(x) \subset V\}$ is open in X for every open subset V of Y. A correspondence $F: X \rightsquigarrow Y$ is said to be *lower semi-continuous* (in short, l.s.c.) if the set $\{x \in X : F(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y. A correspondence $F: X \rightsquigarrow Y$ is said to be *continuous* if it is both u.s.c. and l.s.c. A correspondence $F: X \rightsquigarrow Y$ is said to have *open lower sections* if the set $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for every $y \in Y$. A correspondence $F: X \rightsquigarrow Y$ is said to have *open upper sections* if, for every $x \in X$ $F(x)$ is open in Y. A correspondence $F: X \rightsquigarrow Y$ is said to be *closed* if its graph $\{(x, y) \in X \times Y : y \in F(x)\}$ is closed in $X \times Y$. A correspondence $F: X \rightsquigarrow Y$ is said to be *open* if its graph $\{(x, y) \in X \times Y : y \in F(x)\}$ is open in $X \times Y$. For a set B, we denote its convex hull, closure, and interior, respectively, by co B, $\overline{B}$ (or cl B), and int B.

Remark 1. It is easy to see that if a correspondence F has an open graph then F has open upper and lower sections and if F has open lower sections then it is l.s.c., and the converse statements may not be true.

Let X be a topological space. A function $f: X \to \mathbf{R} \cup \{\pm\infty\}$ is said to be lower semi-continuous (in short, l.s.c.) on X if for each point $x' \in X$, we have

$$\liminf_{x \to x'} f(x) \geq f(x'),$$

or equivalently, its epigraph $\operatorname{epi} f \equiv \{(x, a) \in X \times \mathbf{R} : f(x) \leq a\}$ is a closed subset of $X \times \mathbf{R}$.

DEFINITION 1. Let I be the set of agents. For each agent $i \in I$, let X_i be the strategy space, $D_i \subset X_i$ the feasible strategy set. Denote $X = \prod_{i \in I} X_i$ and let $A_i: X \rightsquigarrow D_i$ be the admissible strategy correspondence and $P_i: X \rightsquigarrow X_i$ the preference correspondence. An *abstract economy* $\Gamma = (X_i, D_i, A_i, P_i)_{i \in I}$ is defined as a family of ordered quarternions (X_i, D_i, A_i, P_i) (triples (X_i, A_i, P_i) when $X_i \equiv D_i$). For each $x \in X$, denote x_i the *i-coordinate* of x (or projection of x in X_i). An *equilibrium* for the abstract economy Γ is an $x^* \in X$ such that for each $i \in I$:

(i) $x_i^* \in \operatorname{cl} A_i(x^*)$ (feasibility),

(ii) $P_i(x^*) \cap A_i(x^*) = \varnothing$ (constrained maximality).

An abstract economy is clearly a composition of a fixed-point theorem and a maximization theorem. Let I be a singleton and get rid of the subscripts. When taking $P(x) = \varnothing$ for all $x \in X$, an abstract economy is simply a fixed-point theorem. While taking $A(x) = X$ for all $x \in X$, an abstract economy becomes a maximization theorem. This is the merit of the setting of an abstract economy. Unfortunately, the fact is that for most cases in the literature once an existence theorem for abstract economy has been established, its merit then gets lost; it is no longer a valuable existence result for fixed-point theorem or maximization theorem, because the assumptions posed on the results are too restrictive. Tulcea's result is exceptional. It is an existence result for abstract economy while retaining its merit as a valuable existence result for fixed-point theorem and maximization theorem. Therefore we will keep Tulcea's framework in our work.

3. THE MAIN RESULTS

In 1975, Shafer and Sonnenschein generalized Debreu's classical existence result for social equilibrium [9] and established the following existence result for abstract economy in finite-dimensional space.

THEOREM 1 [20]. *The abstract economy* $\Gamma = (X_i, A_i, P_i)_{i\in I}$ *admits an equilibrium if for each* $i \in I$:

(a) X_i *is a nonempty, compact, convex subset of* $\mathbf{R}^n$;

(b) $\operatorname{cl} A_i$ *is* u.s.c. *and* A_i *is* l.s.c. *with nonempty convex values*;

(c) P_i *has open graph such that* $x_i \notin \operatorname{co} P_i(x)$ *for all* $x \in X$.

Yannelis and Prabhakar extended Shafer and Sonnenschein's result to infinite-dimensional space in 1983. They proved:

THEOREM 2 [23]. *The abstract economy* $\Gamma = (X_i, A_i, P_i)_{i\in I}$ *admits an equilibrium if for each* $i \in I$:

(a) X_i *is a nonempty, compact, convex, metrizable subset of a locally convex linear topological space* Y_i;

(b) $\operatorname{cl} A_i$ *is* u.s.c. *and* A_i *has open lower sections with nonempty, convex values*;

(c) P_i *has open lower sections such that* $x_i \notin \operatorname{co} P_i(x)$ *for all* $x \in X$.

DEFINITION 2. A correspondence $G_i: X = \prod_{i\in I} X_i \rightsquigarrow X_i$ is said to be of *class* $\mathscr{L}$ (or an $\mathscr{L}$-correspondence) (see Definition 5.1 in [23]), if

(1) $G_i(x)$ is convex in X_i for each $x \in X$,

(2) G_i has open lower sections,

(3) $x_i \notin G_i(x)$ for each $x \in X$.

A correspondence $F_i: X \rightsquigarrow X_i$ is said to be *locally* $\mathscr{L}$*-majorized* (called $\mathscr{C}_i$-majorized in [22] and called $\mathscr{L}^*$-majorized in [10]), if for each $x \in X$, there exist a neighborhood $\mathscr{N}(x)$ of x and an $\mathscr{L}$-correspondence $G_i^x: X \rightsquigarrow X_i$ such that $F_i(y) \subset G_i^x(y)$ for all $y \in \mathscr{N}(x)$. A correspondence $F_i: X \rightsquigarrow X_i$ is said to be $\mathscr{L}$-majorized [23], if there is an $\mathscr{L}$-correspondence $G_i: X \rightsquigarrow X_i$ such that $F_i(y) \subset G_i(y)$ for all $y \in X$.

Since it is known that even a continuous correspondence may fail to have open lower sections, Yannelis and Prabhakar's result can not cover Debreu's classical existence result for social equilibrium [9]. So in 1988 Tulcea proved:

THEOREM 3 [22]. *The abstract economy* $\Gamma = (X_i, A_i, P_i)_{i\in I}$ *admits an equilibrium if for each* $i \in I$:

(a) X_i *is a nonempty, compact, convex subset of a locally convex topological vector space* Y_i;

(b) $\operatorname{cl} A_i$ *is* u.s.c. *with nonempty, convex values*;

(c) P_i *is l.s.c. and locally* $\mathscr{L}$*-majorized*;

(d) *the set* $U_i = \{x \in X: (A_i \cap P_i)(x) \neq \varnothing\}$ *is open.*

Let us compare Tulcea's result with Yannelis and Prabhakar's. We can see that Tulcea extracted Assumption (d) from Assumptions (b) and (c). It is this feature that made Tulcea's result an excellent existence theorem for abstract economy while reserving its values in fixed point theory and maximization theory. Note that the compactness assumption is crucial in the proof of Tulcea's theorem. While in applications, in particular in infinite-dimensional spaces, the compactness is a very restrictive assumption. Therefore in order to relax the compactness assumption and the assumption (c) in the above theorem, Ding *et al.* proved the following result in 1992. While the cost paid is also high.

THEOREM 4 [10]. *The abstract economy* $\Gamma = (X_i, D_i, A_i, P_i)_{i\in I}$ *admits an equilibrium if for each* $i \in I$:

(a) X_i *is a nonempty, convex subset of a locally convex topological vector space* Y_i *and* $D_i \subset X_i$ *is compact*;

(b) A_i: $X \rightsquigarrow D_i$ *has open lower sections with nonempty, convex values and* $\mathrm{cl}\, A_i$ *is* u.s.c.;

(c) $A_i \cap P_i$ *is locally* $\mathcal{L}$*-majorized*;

(d) *The set* $U_i = \{x \in X: (A_i \cap P_i)(x) \neq \emptyset\}$ *is open*;

(e) *There exists a nonempty set* $K_i \subset D_i$ *with* $coK_i \subset D_i$ *such that for each* $x \in U_i \cap E_i$, $(A_i \cap P_i)(x) \cap K_i \neq \emptyset$ *and for each* $x \in X\backslash E_i$, $coA_i(x) \cap K_i \neq \emptyset$, *where* $E_i = \{x \in X: x_i \in \mathrm{cl}\, A_i(x)\}$.

In this paper, we will:

(1) modify Tulcea's result—Theorem 3, by relaxing Assumption (c) that P_i is l.s.c. and locally $\mathcal{L}$-majorized;

(2) generalize the Ding *et al.* result—Theorem 4 by relaxing the condition that A_i: $X \rightsquigarrow D_i$ has to have open lower sections and dropping Assumption (d);

(3) extend Shafer and Sonnenschein's result—Theorem 1 by weakening the assumption that P_i has open graph to that P_i is l.s.c. with open values;

(4) show that when the strategy spaces are metrizable, Theorems 5 and 6 are strict generalizations of the results of Shafer and Sonnenschein, Yannelis and Prabhakar, Tulcea, and Ding *et al.*, respectively;

(5) show that many different approaches in quasi-variational inequalities can be unified by our results.

The approach we adopted in proving these theorems is an extension of Yannelis and Prabhakar's argument in [23].

DEFINITION 3. Let $U_i \subset X$ be a subset. A correspondence A_i: $X \rightsquigarrow X_i$ is said to have (ε-CS)-property on U_i, if for each convex balanced neighborhood V of θ in Y_i, there exists a continuous function f_i^V: $U_i \to X_i$ such that $x_i \neq f_i^V(x)$ and $f_i^V(x) \in A_i(x) + V$ for all $x \in U_i$.

THEOREM 5. *The abstract economy* $\Gamma = (X_i, A_i, P_i)_{i\in I}$ *admits an equilibrium if for each* $i \in I$:

(a) X_i *is a nonempty, compact, convex subset of a locally convex topological vector space* Y_i,

(b) cl A_i *is* u.s.c. *with nonempty, convex values,*

(c) A_i *has* (ε-*CS*)*-property on an open set* U_i *containing the set* $\{x \in X\colon (A_i \cap P_i)(x) \neq \varnothing\}$.

Proof. Let $\{v_{ij}\}_{j\in\mathcal{J}}$ be a fundamental system of convex-balanced neighborhoods of $\theta \in Y_i$, indexed by a filtering set $\mathcal{J}$. Due to (ε-CS)-property, for each $v_{ij}, j \in \mathcal{J}$, there exists a continuous function f_{ij}: $U_i \to X_i$ such that for each $x \in U_i$, $x_i \neq f_{ij}(x)$ and $f_{ij}(x) \in (A_i(x) + v_{ij}) \cap X_i$. Let $\{f_{ij}\}_{j\in\mathcal{J}}$ be the family of such functions. Define a correspondence G_{ij}: $X \rightsquigarrow X_i$ for each $x \in X$ by

$$G_{ij}(x) = \begin{cases} \{f_{ij}(x)\} & \text{if } x \in U_i \\ (\text{cl } A_i(x) + \bar{v}_{ij}) \cap X_i & \text{otherwise.} \end{cases} \tag{3.1}$$

First, we show that the correspondence defined by (cl $A_i(x) + \bar{v}_{ij}) \cap X_i$ is u.s.c. Let $\mathcal{U}_i$ be an open set in Y_i and (cl $A_i(x') + \bar{v}_{ij}) \cap X_i \subset \mathcal{U}_i$. Since (cl $A_i(x') + \bar{v}_{ij}) \cap X_i$ is compact, there exists a convex balanced neighborhood v_1 of θ in Y_i such that

$$[(\text{cl } A_i(x') + \bar{v}_{ij}) \cap X_i] + v_1 \subset \mathcal{U}_i.$$

In view of Lemma 1 in [12], there exists another convex balanced neighborhood v_2 of θ in Y_i such that

$$(\text{cl } A_i(x') + \bar{v}_{ij} + v_2) \cap (X_i + v_2) \subset [(\text{cl } A_i(x') + \bar{v}_{ij}) \cap X_i] + v_1 \subset \mathcal{U}_i.$$

Since cl A_i is assumed to be u.s.c., there exists a neighborhood $N(x')$ of x' such that for each $x \in N(x')$

$$\text{cl } A_i(x) \subset (\text{cl } A_i(x') + v_2) \cap X_i.$$

It follows that for each $x \in N(x')$

$$(\text{cl } A_i(x) + \bar{v}_{ij}) \cap X_i \subset (\text{cl } A_i(x') + \bar{v}_{ij} + v_2) \cap (X_i + v_2) \subset \mathcal{U}_i.$$

So $(\mathrm{cl}\, A_i(x) + \bar{v}_{ij}) \cap X_i$ is u.s.c. Next we show that the correspondence $G_{ij}: X \rightsquigarrow X_i$ is u.s.c. with closed convex values. As for each open set $V_i \subset X_i$, the set

$$\{x \in X: G_{ij}(x) \subset V_i\} = \{x \in X: [(\mathrm{cl}\, A_i(x) + \bar{v}_{ij}) \cap X_i] \subset V_i\} \\ \cup \{x \in U_i: f_{ij}(x) \in V_i\}$$

is open, because $(\mathrm{cl}\, A_i(x) + \bar{v}_{ij}) \cap X_i$ is u.s.c., U_i is open, f_{ij} is continuous and satisfies $f_{ij}(x) \in (\mathrm{cl}\, A_i(x) + \bar{v}_{ij}) \cap X_i$ for all $x \in U_i$.

Next we define a correspondence $G_j: X \rightsquigarrow X$ for each $x \in X$ by

$$G_j(x) = \prod_{i \in I} G_{ij}(x).$$

By Lemma 3 in [12, p. 124], the correspondence G_j is u.s.c. with closed convex values from a compact set X (by Tychonoff's theorem) into itself. By Theorem 1 in [12, p. 122], there exists a point $^*x^j \in X$ such that $^*x^j \in G_j(^*x^j)$. Observe that if $^*x^j \in U_i$ for some $i \in I$, it leads to $^*x^j_i = f_{ij}(^*x^j)$, a contradiction to our assumption that $x_i \neq f_{ij}(x)$ for all $x \in U_i$. So we get $^*x^j \in X \backslash U_i$ for all $i \in I$, or $^*x^j \in X \backslash U_{i \in I}\, U_i$ and also $^*x^j_i \in (\mathrm{cl}\, A_i(^*x^j) + \bar{v}_{ij}) \cap X_i$, $\forall i \in I$. Since U_i is open in the compact set X, the set $X \backslash U_{i \in I}\, U_i$ is compact. Therefore there exist an ultrafilter $\mathcal{U}$ finer than the filter of sections of $\mathcal{J}$ and a point $x^* \in X \backslash \bigcup_{i \in I} U_i$ such that

$$x^* = \lim_{(j, \mathcal{U})} {}^*x^j. \tag{3.2}$$

This implies

$$A_i(x^*) \cap P_i(x^*) = \varnothing, \qquad \forall i \in I \tag{3.3}$$

and also means

$$x_i^* = \lim_{(j, \mathcal{U})} {}^*x_i^j, \qquad \forall i \in I. \tag{3.4}$$

Now to prove the theorem it remains to show that for each $i \in I$, $x_i^* \in \mathrm{cl}\, A_i(x^*)$. Let V_i be any convex balanced neighborhood of θ in Y_i. Since $\mathrm{cl}\, A_i(x^*)$ is a compact set contained in the open set $\mathrm{cl}\, A_i(x^*) + V_i$, there exist two convex balanced neighborhoods V_1 and V_2 of θ in Y_i such that

$$\mathrm{cl}\, A_i(x^*) + V_1 + V_2 \subset \mathrm{cl}\, A_i(x^*) + V_i.$$

The correspondence cl A_i is assumed to be u.s.c., hence the set

$$N(x^*) = \{x \in X : \mathrm{cl}\, A_i(x) \subset \mathrm{cl}\, A_i(x^*) + V_1\}$$

is an open set containing the point x^*. By (3.2), there exists $j_0 \in \mathcal{U}$ such that $^*x^j \in N(x^*)$ whenever $j \geqslant j_0$ and $j \in \mathcal{U}$, where $\geqslant$ is the order relation in the filtering set $\mathcal{J}$. In view of the fact that $\mathcal{U}$ is an ultrafilter finer than the filter of sections of $\mathcal{J}$, there exists $j_1 \in \mathcal{U}$ such that $j \geqslant j_1$ and $j \in \mathcal{U}$ imply $\bar{v}_{ij} \subset V_2$. Therefore for $j \geqslant \max(j_0, j_1)$ and $j \in \mathcal{U}$, we have

$$^*x_i^j \in \mathrm{cl}\, A_i(^*x^j) + \bar{v}_{ij} \subset \mathrm{cl}\, A_i(x^*) + V_1 + V_2 \subset \mathrm{cl}\, A_i(x^*) + V_i.$$

It then follows from (3.4) that

$$x_i^* \in \mathrm{cl}\, A_i(x^*) + \overline{V}_i, \qquad \forall i \in I$$

holds for any convex balanced neighborhood V_i of θ in Y_i. Therefore we obtain

$$x_i^* \in \mathrm{cl}\, A_i(x^*), \qquad \forall i \in I$$

and complete the proof. ∎

Remark 2. *If I is a singleton, we can write $X = X_i$, $P = P_i$, $A = A_i$ and $U = U_i$.*

(1) When $P(x) = \varnothing$ for all $x \in X$, Assumption (c) in Theorem 5 is trivially satisfied, so it is precisely Kakutani and Fan's fixed point theorem for correspondence.

(2) When $A(x) = X$ for all $x \in X$, Theorem 5 is a generalization of many existence theorems for maximal elements (e.g., Theorem 5.1 and Corollary 5.1 in [23]). Observe that if f: $U \to X$ is a continuous function. If $U = X$ then by Brouwer's fixed point theorem, Assumption (c) will not be satisfied. So if Assumption (c) is satisfied then $U \neq X$. Any $x \in X \setminus U$ is a maximal element for the preference P.

It is clear that the compactness assumption for X_i is also crucial in the above proof. It cannot be simply dropped without adding other conditions. The cost for relaxing compact assumption on X_i sometimes is very high. For an example, in Theorem 4 (Ding *et al.*), when X_i is not compact, for an abstract economy to admit an equilibrium, the correspondence A_i has to have open lower sections in addition to the extra assumption (e). So result of Ding *et al.* cannot cover Debreu's classical existence result for social equilibrium [9]. On the other hand, let us observe that when I is a singleton and we drop all subscripts, if we let $P(x) = \varnothing$ for all $x \in X$ and compare Theorem 4 with Himmelberg's fixed point theorem [13, Theorem 2] (a generalization of Kakutani and Fan's fixed point theorem on noncompact set), we can see that the assumption that A has open lower sections is an extra condition. So the result is no longer valuable in fixed point theory. Our next main result is to relax the compactness assumption on X_i, while retaining the merit of the setting of abstract economies—to be a valuable result in both fixed point theory and the theory for maximization with respect to preference relations. We will show that the condition that A_i has open lower sections can be dropped. To do this we need to introduce the following.

DEFINITION 4. Let $D_i \subset X_i$ and $U_i \subset X$. A correspondence $A_i: X \rightsquigarrow D_i$ is said to have (CS)-property on U_i, if there exists a continuous function $f_i: U_i \to D_i$ such that $x_i \neq f_i(x)$ and $f_i(x) \in \operatorname{cl} A_i(x)$ for all $x \in U_i$.

THEOREM 6. *The abstract economy* $\Gamma = (X_i, D_i, A_i, P_i)_{i \in I}$ *admits an equilibrium if for each* $i \in I$:

(a) *X_i is a nonempty, convex subset of a locally convex topological vector space Y_i and $D_i \subset X_i$ is compact*;

(b) $\operatorname{cl} A_i: X \rightsquigarrow D_i$ *is* u.s.c. *with nonempty, convex values*;

(c) *A_i has (CS)- property on an open set $U_i \subset \operatorname{co} D = \prod_{i \in I} \operatorname{co} D_i$ containing the set* $\{x \in \operatorname{co} D: (A_i \cap P_i)(x) \neq \varnothing\}$.

Proof. For each $i \in I$, let $f_i: U_i \to D_i$ be a continuous function defined in (CS)-property such that $x_i \neq f_i(x)$ and $f_i(x) \in \operatorname{cl} A_i(x)$ for all $x \in U_i$. Define a correspondence $G_i: X \rightsquigarrow D_i$ for each $x \in X$ by

$$G_i(x) = \begin{cases} \{f_i(x)\} & \text{if } x \in U_i \\ \operatorname{cl} A_i(x) & \text{otherwise.} \end{cases} \tag{3.5}$$

Notice that if $U_i = \varnothing$ then $G_i \equiv \operatorname{cl} A_i$. Also G_i is u.s.c., as for each open set $V_i \subset X_i$, the set

$$\{x \in X : G_i(x) \subset V_i\} = \{x \in X : \operatorname{cl} A_i(x) \subset V_i\} \cup \{x \in U : f_i(x) \in V_i\}$$

is open in X_i, because U_i is open, $\operatorname{cl} A_i: X \rightsquigarrow D_i$ is u.s.c. and $f_i(x) \in \phi(x) \subset \operatorname{cl} A_i(x)$ $\forall x \in U_i$. Let $D = \prod_{i \in I} D_i$. Then by Tychonoff's theorem, D is compact in the convex set X. Define $G: X \rightsquigarrow D$ for each $x \in X$ by $G(x) = \prod_{i \in I} G_i(x)$. By Lemma 3 in [12, p. 124], G is u.s.c. with nonempty, convex, and closed values. By a generalization of Kakutani and Fan's fixed point theorem due to Himmelberg [13, Theorem 2]), there exists a point $x^* \in X$ such that $x^* \in G(x^*)$. Note that $x^* \in U_i$ implies $x_i^* =$

$G_i(x^*) = f_i(x^*)$ and leads to a contradiction to (CS)-property that $x_i \neq f_i(x)$ $\forall x \in U_i$. Hence, $x^* \notin U_i$ for all $i \in I$. This implies that $x_i^* \in \operatorname{cl} A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \varnothing$. The proof is complete. ∎

Remark 3. If I is a singleton, we can write $X = X_i$, $D_i = D$, $P = P_i$, $A = A_i$ and $U = U_i$.

(1) When $P(x) = \varnothing$ for all $x \in X$, Assumption (c) in Theorem 6 is trivially satisfied, so it is precisely Himmelberg's fixed point theorem for correspondence.

(2) When $A(x) = X$ for all $x \in X$, Theorem 6 becomes an existence theorem for maximal elements with respect to the preference correspondence P on a noncompact set X. Observe that if $f: U \to X$ is a continuous function then it is also an u.s.c. (single-valued) correspondence. When $U = X$ Himmelberg's fixed point theorem for correspondence assures that there is $x^* \in D$ such that $x^* = f(x^*)$. So Assumption (c) will not be satisfied. Therefore if Assumption (c) is satisfied then $U \neq X$. Any $x \in X-U$ is a maximal element for the preference P.

When the space X_i is finite dimensional, Remark 6.3. in [24] claims that Assumption (c) in Theorem 1 cannot be relaxed as that $P_i(x)$ has open lower and upper sections. In contrast, however, we will show that Assumption (c) in Theorem 1 can be relaxed as that $P_i(x)$ has open lower and upper sections and even weaker. This is stated in the next theorem as our third main result, which is a strict generalization of Shafer and Sonnenschein's existence theorem (Theorem 1) for abstract economy in finite-dimensional space and is not covered by any result in the literature. To prove this we need to establish several propositions that are of independent interest.

THEOREM 7. *The abstract economy* $\Gamma = (X_i, D_i, A_i, P_i)_{i\in I}$ *admits an equilibrium if for each* $i \in I$:

(a) X_i *is a nonempty, convex subset of a locally convex topological vector space* Y_i *and* $D_i \subset X_i$ *is compact and finite-dimensional*;

(b) $A_i: X \rightsquigarrow D_i$ *is* l.s.c. *with nonempty convex values such that* $\operatorname{cl} A_i$ *is* u.s.c.;

(c) P_i *is* l.s.c. *with open values such that* $x_i \notin \operatorname{co} P_i(x)$ *for all* $x \in X$.

Before we prove the theorem, let us observe the assumptions in Tulcea's result—Theorem 3. Under Assumption (c) that P_i is l.s.c. and locally $\mathcal{L}$-majorized (which implies $x_i \notin \operatorname{co} P_i(x)$ for all $x \in X_i$), it follows that co P_i is l.s.c. and locally $\mathcal{L}$-majorized. Since the correspondence co P_i is convex-valued, it may have better properties than P_i does. So a slight generalization of Tulcea's result can be established as follows:

THEOREM 8. *The abstract economy* $\Gamma = (X_i, A_i, P_i)_{i \in I}$ *admits an equilibrium if for each* $i \in I$:

(a) X_i *is nonempty, compact, convex subset of a locally convex topological vector space* Y_i;

(b) $\operatorname{cl} A_i$ *is* u.s.c. *with nonempty, convex values*;

(c) P_i *is* l.s.c. *and locally* $\mathcal{L}$-*majorized*;

(d) *either the set* $U_i = \{x \in X : (A_i \cap P_i)(x) \neq \varnothing\}$ *or the set* $U_i' = \{x \in X : (A_i \cap \operatorname{co} P_i)(x) \neq \varnothing\}$ *is open.*

Or similarly we can replace all P_i by $\operatorname{co} P_i$ in Theorem 3. This slight generalization is necessary, because in some application the set $\{x \in X : (A_i \cap P_i)(x) \neq \varnothing\}$ is not open while the set $\{x \in X : (A_i \cap \operatorname{co} P_i)(x) \neq \varnothing\}$ is open. We will provide an example to show this after we prove several technical lemmas.

PROPOSITION 1. *Let* X *be a topological space and* $G: X \rightsquigarrow \mathbf{R}^n$ *be a convex valued correspondence* l.s.c. *at* $x_0 \in X$. *Then for any compact set* $C \subset \operatorname{int} G(x_0)$, *there exists a neighborhood* $\mathcal{N}(x_0)$ *of* x_0 *such that* $C \subset G(x)$ $\forall x \in \mathcal{N}(x_0)$.

Proof. Let $y_0 = 0 \in \mathbf{R}^n$ and $B(y_0, r)$ be the open ball of radius r centered at y_0. For $j = 2i$ or $j = 2i - 1$, let $z_j = (0, \ldots, \pm\frac{3}{4}r, 0, ..0)$ have only nonzero value at i-th component, where "+" is for $j = 2i$ and "−" is for $j = 2i - 1$. Then choose any one point $y_j \in B(z_j, \frac{1}{4}r)$, $(j = 1, 2, \ldots, 2n)$, we have

$$B(y_0, r_0) \subset \operatorname{co}\{y_1, y_2, \ldots, y_{2n}\}, \tag{3.6}$$

where $r_0 = r/(2\sqrt{n})$. Now we prove that for each $y_0 \in \operatorname{int} G(x_0)$, there exist $r_0 > 0$ and neighborhood $\mathcal{N}(x_0)$ such that

$$B(y_0, r_0) \subset G(x), \quad \forall x \in \mathcal{N}(x_0).$$

For $y_0 \in \operatorname{int} G(x_0)$, there exists $B(y_0, r)$ such that $B(y_0, r) \subset G(x_0)$. By the fact (3.6) we mentioned above, there exist $z_j \in B(y_0, r)$, $(j = 1, 2, .., 2n)$ such that for each $y_j \in B(z_j, \frac{1}{4}r)$, we have

$$B(y_0, r_0) \subset \operatorname{co}\{y_1, y_2, \ldots, y_{2n}\}$$

for $r_0 = r/(2\sqrt{n})$. Now for the above $2n$ open sets $B(z_j, \frac{1}{4}r)$ in $G(x_0)$, since $G(x)$ is l.s.c. at x_0, there exists a neighborhood $\mathcal{N}(x_0)$ of x_0 such that

$$G(x) \cap B(z_j, \tfrac{1}{4}r) \neq \varnothing, j = 1, 2, \ldots, 2n, \qquad \forall x \in \mathcal{N}(x_0).$$

Fix $x \in \mathcal{N}(x_0)$. For each $j = 1, 2, \ldots, 2n$, choosing $y_{xj} \in G(x) \cap B(z_j, \frac{1}{4}r)$ and taking the convexity of $G(x)$ into account, we have

$$B(y_0, r_0) \subset \text{co}\{y_{x1}, y_{x2}, \ldots, y_{x2n}\} \subset G(x), \qquad \forall x \in \mathcal{N}(x_0).$$

Let C be any compact set in int $G(x_0)$. For each $y \in C$, there exist a ball $B(y, r_y)$ and a neighborhood $\mathcal{N}_y(x_0)$ such that $B(y, r_y) \in G(x)$, $\forall x \in \mathcal{N}_y(x)$. Because C is compact and $C \subset \bigcup_{y \in C} B(y, r_y)$, there exist $\{y_1, y_2, \ldots, y_m\} \subset C$ such that $C \subset \bigcup_{j=1}^m B(y_j, r_{y_j})$. Take $\mathcal{N}(x_0) = \bigcup_{j=1}^m \mathcal{N}_{y_j}(x_0)$, it yields

$$C \subset \bigcup_{j=1}^m B(y_j, r_{y_j}) \subset G(x), \qquad \forall x \in \mathcal{N}(x_0). \quad \blacksquare$$

With the above proposition, it is straightforward to show:

PROPOSITION 2.[1] *Let X be a topological space and $G\colon X \rightsquigarrow \mathbf{R}^n$ be a convex-valued correspondence. Then G has an open graph in $X \times \mathbf{R}^n$ if and only if G is* l.s.c. *and open-valued in X.*

As a by-product, the following lemma is also of interest.

PROPOSITION 3. *Let $\mathbf{X}$ be a topological space and $f(x, y)$ be a functional on $X \times \mathbf{R}^n$ such that for each fixed $x \in \mathbf{X}$, $f(x, \cdot)$ is quasiconcave. Then f is* l.s.c. *if and only if f is* l.s.c. *in each variable.*

Proof. For each $r \in R$, define a correspondence $G\colon X \rightsquigarrow \mathbf{R}^n$ by

$$G_r(x) = \{y \in \mathbf{R}^n : f(x, y) > r\}.$$

Since $f(x, y)$ is quasiconcave in y, G_r is convex-valued. Then f is l.s.c. if and only if G_r has an open graph and by Proposition 2, G_r has an open graph if and only if G_r has open lower sections and upper sections. The last is true if and only if f is l.s.c. in each variable. $\blacksquare$

In Lemma 1, the set C can not be weakened as either bounded or closed. This can be seen in the following examples.

EXAMPLE 1. Let $G\colon [-1, 1] \rightsquigarrow \mathbf{R}^2$ be defined for each $z \in [-1, 1]$ by

$$G(z) = B((z, 0), 1).$$

It is easy to see that G is convex open-valued and has an open graph, therefore it is l.s.c. But for the bounded subset $B((0, 0), 1) \subset G(0)$, no other values $G(x)$ can contain $B((0, 0), 1)$.

[1] The author wishes to acknowledge several discussions with Dr. G. Tian.

EXAMPLE 2. Let G: $[-1, 1] \rightsquigarrow \mathbf{R}^2$ be defined for each $z \in [-1, 1]$ by

$$G(z) = \begin{cases} \{(x,y) : -1 < x < \frac{1}{|z|}, -1 < y < 1\}, & \text{if } z \neq 0, \\ \{(x,y) : -1 < x < \infty, -1 < y < 1\}, & \text{if } z = 0. \end{cases}$$

Also it is clear that G is convex open-valued and has an open graph, therefore it is l.s.c. But for the closed set $\{(x, 0) : 0 \leq x\} \subset G(0)$, no other values $G(x)$ can contain it.

LEMMA 1. (Proposition 2.6., [16]). *If X is a topological space, Y is a topological vector space and if P: $X \rightsquigarrow Y$ is* l.s.c. *then $P'(x) = \mathrm{co}\, P(x)$ is also* l.s.c.

With the above propositions, we can now prove Theorem 7.

Proof of Theorem 7. By Lemma 1 and Proposition 2, we conclude that co $P_i(x)$ has open graph in $X \times X_i$. The correspondence F_i: $X \rightsquigarrow X_i$ defined by $F_i(x) = A_i(x) \cap \mathrm{co}\, P_i(x)$ for all $x \in X$ is l.s.c. due to $U_i = \{x \in X : (A_i \cap \mathrm{co}\, P_i)(x) \neq \emptyset\}$ is open. The conclusion then follows from Theorem 8. ∎

Remark 4. (1) In Theorem 7, the set $U_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ may fail to be open, Yannelis provided such example in [24, p. 108]. So Tulcea's result—Theorem 3 cannot be applied directly.

(2) Assumption (c) in Theorem 7 by no means implies that $P_i(x)$ has an open graph, simply because that $P_i(x)$ is not necessarily a convex-valued correspondence.

Therefore Theorem 7 is a nontrivial generalization of result in [20] (also see [24]).

4. SOME COROLLARIES

In this section, we show that when X_i is also metrizable, as the cases in most applications, Theorems 5 and 6 are, respectively, strict generalization of Theorem 3 (Tulcea) and Theorem 4 (Ding *et al.*) Compactness assumption in Theorem 1 (Shafer and Sonnenschein) and Theorem 2 (Yannelis and Prabhakar) can be simply relaxed. First we indicate that the condition that P_i is l.s.c. in Theorem 3 has been dropped in the following corollary.

COROLLARY 1. *If for each $i \in I$:*

(a) *X_i is a nonempty, compact, convex metrizable subset of a locally convex topological vector space Y_i,*

(b) cl A_i *is* u.s.c. *with nonempty, convex values,*

(c) P_i *is locally* $\mathscr{L}$*-majorized,*

(d) *the set* $U_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ *is open,*

then A_i *has* (ε-*CS*)-*Property and thus the abstract economy* $\Gamma = (X_i, A_i, P_i)_{i \in I}$ *admits an equilibrium.*

Proof. We only need to show that A_i has (ε-CS)-Property and then apply Theorem 5. By Theorem 3 in [22], there is a nonempty filtering set $\mathscr{J}$ with ordering $\preccurlyeq$ and for each $i \in I$, A_i has an *upper approximation* of family $\{A_{ij}\}_{j \in \mathscr{J}}$ of open correspondences from X to X_i index by $\mathscr{J}$ (we only need that A_{ij} has open lower sections) such that (1) $A_i \subset A_{ij}$ for every $j \in \mathscr{J}$; (2) for every $j \in \mathscr{J}$ there is $j^* \in \mathscr{J}$ such that $A_{ih} \subset A_{ij}$ if $h \in \mathscr{J}$ and $j^* \preccurlyeq h$; (3) for every $x \in X$ and $V \in \{v_{ij}\}_{j \in \mathscr{J}}$ (the fundamental system of convex-balanced neighborhoods of θ in Y_i), there is $j_{x,V} \in \mathscr{J}$ such that $A_{ih}(x) \subset A_i(x) + V$ if $h \in \mathscr{J}$ and $j_{x,V} \preccurlyeq h$. On the other hand, when X_i is a metrizable set, X is metrizable as well [14, p. 50]; each open set in X is paracompact [16, p. 381]. Then by Lemma 1 in [10], P_i is majorized by an $\mathscr{L}$-correspondence ϕ_i: $U_i \rightsquigarrow X_i$ (with open lower section and convex values such that $x_i \notin \phi_i(x)$ for each $x \in U_i$). Define a correspondence ψ_j^i: $U_i \rightsquigarrow X_i$ for each $x \in U_i$ by

$$\psi_j^i(x) = A_{ij}(x) \cap \phi_i(x) \quad \forall j \in \mathscr{J}.$$

Then ψ_j^i is a nonempty convex-valued correspondence with open lower sections. Since U_i is open in X, it is paracompact, by Theorem 3.1 in [23] ψ_j^i admits a continuous selection f_j^i: $U_i \to X_i$. It follows that

$$f_j^i(x) \in \psi_j^i(x) = A_{ij}(x) \cap \phi_i(x) \text{ and } x_i \notin \phi_i(x) \quad x \in U_i,$$

or we have

$$f_j^i(x) \in A_i(x) + v_{ij} \text{ and } x_i \neq f_j^i(x) \quad \forall x \in U_i,$$

where $v_{ij} \in \{v_{ij}\}_{j \in \mathscr{J}}$. We conclude that A_i has (ε-CS)-property on U_i and then the conclusion follows from Theorem 5. ∎

When X_i is also metrizable, Assumption (e) in Theorem 4 can be simply dropped as shown in the following.

COROLLARY 2. *If for each* $i \in I$:

(a) X_i *is a nonempty, convex metrizable subset of a locally convex topological vector space* Y_i *and* $D_i \subset X_i$ *is compact,*

(b) A_i: $X \rightsquigarrow D_i$ *has open lower sections with nonempty, convex values and* cl A_i *is* u.s.c.,

(c) *$A_i \cap P_i$ is locally* $\mathscr{L}$-majorized,

(d) *the set $U_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open,*

then A_i has (CS)-property on U_i and thus the abstract economy $\Gamma = (X_i, D_i, A_i, P_i)_{i \in I}$ admits an equilibrium.

Proof. We only need to show that A_i has (CS)-property on U_i and apply Theorem 6. Since X_i is also metrizable, by Lemma 1 in [10], P_i is majorized by an $\mathscr{L}$-correspondence ϕ_i: $U_i \rightsquigarrow X_i$ (with open lower section and convex values such that $x_i \notin \phi_i(x)$ for each $x \in U_i$). Define a correspondence ψ_i: $U_i \rightsquigarrow X_i$ for each $x \in U_i$ by

$$\psi_i(x) = A_i(x) \cap \phi_i(x).$$

Then ψ_i is a nonempty convex-valued correspondence with open lower sections. Since U_i is open in X, it is paracompact, by Theorem 3.1 in [23] ψ_i admits a continuous selection f_i: $U_i \rightarrow X_i$. It follows that

$$f_i(x) \in \psi_i(x) = A_i(x) \cap \phi_i(x) \text{ and } x_i \notin \phi_i(x) \quad x \in U_i,$$

or we have

$$f_i(x) \in A_i(x) \text{ and } x_i \neq f_i(x) \quad \forall x \in U_i.$$

We conclude that A_i has (CS)-property on U_i and then the conclusion follows from Theorem 6. ▮

Now let us examine Yannelis and Prabhakar's result, Theorem 2, we see that all the conditions in Corollary 2 are satisfied, so the compactness assumption can be simply relaxed as in Corollary 2. As another corollary of Theorem 6, the next result extends Shafer and Sonnenschein's result to infinite-dimensional spaces, also relaxes the compactness assumption and is not covered by Theorem 3 (Tulcea's) or Theorem 4 (Ding *et al.*).

COROLLARY 3. *If for each $i \in I$:*

(a) *X_i is a nonempty, convex metrizable subset of a locally convex topological vector space Y_i and $D_i \subset X_i$ is compact,*

(b) *A_i: $X \rightsquigarrow D_i$ is* l.s.c. *with nonempty, convex values and* cl A_i *is* u.s.c.,

(c) *P_i: $X \rightsquigarrow D_i$ has open graph such that $x_i \notin$* co *$P_i(x)$ for all $x \in X$,*

then the abstract economy $\Gamma = (X_i, D_i, A_i, P_i)_{i \in I}$ admits an equilibrium.

Proof. We only need to show that A_i has (CS)-property on the set $U_i = \{x \in X | (A_i \cap \text{co } P_i)(x) \neq \emptyset\}$ and U_i is open, and then apply

Theorem 6. By Assumption (b), cl A_i is continuous, and by Assumption (c), co P_i has open graph, thus the correspondence $\phi_i\colon U_i \rightsquigarrow D_i$ defined, for each $x \in U_i$ by $\phi_i(x) = (\text{cl } A_i \cap \text{co } P_i)(x)$ has open lower sections (cf. [24]). It follows that U_i is open in metrizable set X, therefore U_i is paracompact [14, p. 50 and 16, p. 381]. By Theorem 3.1 in [23], ϕ_i admits a continuous selection $f_i\colon U_i \to D_i$. Since $x_i = f(x)$ will lead to $x_i \in \text{co } P_i(x)$, a contradiction to Assumption (c), we have

$$f_i(x) \in \text{cl } A_i(x) \text{ and } x_i \neq f_i(x) \quad \forall x \in U_i,$$

i.e., A_i has (CS)-property on the open set U_i and then the conclusion follows from Theorem 6. ∎

5. Quasi-Variational Inequalities

In this section we show that many different approaches in quasi-variational inequalities can be unified by the theoretical framework of abstract economies.

The following existence theorem for quasi-variational inequality is in turn a generalization of many results in the literature.

LEMMA 2 [2]. *Assume that*

(a) *X is a nonempty compact convex set of a locally convex topological vector space,*

(b) *$A\colon X \rightsquigarrow X$ is* u.s.c. *with nonempty closed convex values,*

(c) *$\phi\colon X \times X - \mathbf{R} \cup \{\pm\infty\}$ is defined such that $\phi(x, y)$ is* l.s.c. *in x and concave in y with $\phi(x, x) \leq 0$ for all $x \in X$,*

(d) *the set*

$$U = \{x \in X \mid \sup_{y \in A(x)} \phi(x, y) \leq 0\}$$

is closed.

Then there exists a solution $x^* \in X$ to the quasi-variational inequality

$$x^* \in A(x^*), \quad \phi(x^*, \ y) \leq 0, \quad \forall y \in A(x^*). \tag{5.1}$$

For a function $\phi\colon X \times X \to \mathbf{R}$, 0-diagonal concavity (0-DCV) and 0-diagonal quasi-concavity (0-DQCV) have been defined in [25] to study variational and quasi-variational inequalities.

Let X be a convex set of a topological vector space E and let $\phi\colon X \times$

$X \to \mathbf{R} \cup \{\pm\infty\}$ be a functional. The functional $(x, y) \mapsto \phi(x, y)$ is said to be 0-*diagonally concave* (in short, 0-DCV) in y (cf. [25]), if for any finite subset $\{y_1, \ldots, y_m\} \subset X$ and any $y_\lambda \in \operatorname{co}\{y_1, \ldots, y_m\}$ (i.e., $y_\lambda = \sum_{j=1}^m \lambda_j y_j$ for $\lambda_j \geq 0$ with $\sum_{j=1}^m \lambda_j = 1$), we have

$$\sum_{j=1}^{m} \lambda_j \phi(y_\lambda, y_j) \leq 0. \tag{5.2}$$

A function $(x, y) \mapsto \phi(x, y)$ is said to be 0-*diagonally quasi-concave* (in short, 0-DQCV) in y (cf. [25]), if for any finite subset $\{y_1, \ldots, y_m\} \subset X$ and any $y_\lambda \in \operatorname{co}\{y_1, \ldots, y_m\}$,

$$\min_j \phi(y_\lambda, y_j) \leq 0.$$

A functional $(x, y) \mapsto \phi(x, y)$ is said to be 0-*diagonally* (*quasi-*)*convex* (in short, 0-DQCX) in y if $-\phi$ is 0-diagonally (quasi-)concave.

In the above existence result Lemma 2, the condition that ϕ is concave in y with $\phi(x, x) \leq 0$ for all $x \in X$ has been weakend in [25] to that ϕ is 0-DCV and further weakend in [21] to that ϕ is 0-DQCV with certain interior point assumption. This is quite restrictive, because X is sametime a compact set. Now with the theoretical setting of abstract economy we are able to get rid of the interior point assumption and prove the following:

COROLLARY 4. *Assume that*

(a) *X is a nonempty compact convex set of a locally convex topological vector space,*

(b) *$A: X \rightsquigarrow X$ is* u.s.c. *with nonempty closed convex values,*

(c) *$\phi: X \times X \to \mathbf{R} \cup \{\pm\infty\}$ is defined such that $\phi(x, y)$ is* l.s.c. *in x and* 0-DQCV *in y,*

(d) *the set*

$$U = \{x \in X \mid \sup_{y \in A(x)} \phi(x, y) \leq 0\}$$

is closed.
Then there exists a solution $x^* \in X$ to the quasi-variational inequality

$$x^* \in A(x^*), \qquad \sup_{y \in A(x^*)} \phi(x^*, y) \leq 0.$$

Proof. Define a correspondence $P: X \rightsquigarrow X$ for each $x \in X$ by $P(x) = \{y \in X \mid \phi(x, y) > 0\}$. Thus, to show the conclusion of the theorem, it is equivalent to showing that there exists $x^* \in A(x^*)$ such that $A(x^*) \cap P(x^*) = \varnothing$.

By the 0-diagonal quasi-concavity, $x \notin \text{co}\, P(x)$ for all $x \in X$. To see this, suppose, by way of contradiction, that there exists some point $x_\lambda \in X$ such that $x_\lambda \in \text{co}\, P(x_\lambda)$. Then there exist finite points, $x_1, \ldots, x_m$ in X, and $\lambda_j \geq 0$ with $\sum_{j=1}^m \lambda_j = 1$ such that $x_\lambda = \sum_{j=1}^m \lambda_j x_j$ and $x_i \in P(x_\lambda)$ for all $i = 1, \ldots, m$. That is, $\phi(x_\lambda, x_i) > 0$ for all i, which contradicts the hypothesis that $\phi(x, y)$ is 0-DQCV in y. $\phi(x, y)$ is l.s.c. in x implies that P has open lower sections. Assumption (d) is equivalent to that the set $U = \{x \in X | A(x) \cap P(x) \neq \varnothing\}$ is open. The result then follows from Theorem 3. ∎

Why it is necessary to study quasi-variational inequalities where ϕ is 0-DQCV? Because it enables us to unify different approaches in quasi-variational inequalities.

Remark 5. The following two types of quasi-variational inequality problems have wide applications (see [5, 17]).

Let X be a topological vector space with its dual X', $A: X \rightsquigarrow X$ the admissible action correspondence and $f: X \to X'$ a function. A solution to the quasi-variational inequality problem is an $x^* \in X$ such that

$$x^* \in A(x^*) \text{ and } \sup_{y \in A(x^*)} \langle f(y), x^* - y \rangle \leq 0; \tag{T1}$$

$$x^* \in A(x^*) \text{ and } \sup_{y \in A(x^*)} \langle f(x^*), x^* - y \rangle \leq 0; \tag{T2}$$

For (T1), monoton operator f has been introduced in the literature to prove the existence. If we let $\phi(x, y) = \langle f(y), x - y \rangle$ then it has been show in [25] that $\phi(x, y)$ is 0-DCV in y if and only if f is monoton. While for (T2), if we let $\phi(x, y) = \langle f(x), x - y \rangle$ then $\phi(x, y)$ is always 0-DQCV in y. So different approaches for existence of solutions to quasi-variational inequalities can be unified. With the setting of abstract economy we can easily pose conditions to assure the existence of solutions to quasi-variational inequalities of types (T1) and (T2). For an example, we can assume that X is a nonempty compact convex set of a locally convex topological vector space and the set

$$\{(x, y) \in X \times X : \langle f(y), x - y \rangle \leq 0\} \quad \text{(for (T1)-type)} \tag{5.3}$$

or

$$\{(x, y) \in X \times X : \langle f(x), x - y \rangle \leq 0\} \quad \text{(for (T2)-type)} \tag{5.4}$$

is, respectively, closed. (This by no means implies that $f(x)$ is continuous.) and A is continuous with nonempty convex closed values. Then quasi-variational inequality (T1) or (T2), respectively, admits a solution.

For finite-dimensional case, we have the following:

COROLLARY 5. *Let X be a nonempty compact convex set in $\mathbf{R}^n$. Suppose that*

(i) *$X \rightsquigarrow X$ is a continuous correspondence with nonempty closed and convex values,*

(ii) *$\phi: X \times X \to \mathbf{R} > \{\pm\infty\}$ is* l.s.c. *in each variable and* 0*-diagonally quasi-concave in the second variable.*

Then there exists $x^* \in A(x^*)$ such that $\sup_{y \in A(x^*)} \phi(x^*, y) \leq 0$.

Proof. Follows from Theorem 7. ∎

Remark 6. With the help of the last theorem and Proposition 2, it can be shown that when in a finite-dimensional space, the conditions (5.3) and (5.4) in Remark 5 can be further relaxed.

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